

CONVEX REAL PROJECTIVE STRUCTURES AND WEIL'S LOCAL RIGIDITY THEOREM

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ABSTRACT. For an n -dimensional real hyperbolic manifold M , we calculate the Zariski tangent space of a character variety $\chi(\pi_1(M), SL(n+1, \mathbb{R}))$, $n > 2$ at Fuchisan loci to show that the tangent space consists of cubic forms. Furthermore we prove the Weil's local rigidity theorem for uniforml hyperbolic lattices using real projective structures.

1. INTRODUCTION

A flat projective structure on an n -dimensional manifold M is a $(\mathbb{RP}^n, PSL(n+1, \mathbb{R}))$ -structure, i.e., there exists a maximal atlas on M whose transition maps are restrictions to open sets in \mathbb{RP}^n of elements in $PSL(n+1, \mathbb{R})$. Then there exist a natural holonomy map $\rho : \pi_1(M) \rightarrow PSL(n+1, \mathbb{R})$ and a developing map $f : \tilde{M} \rightarrow \mathbb{RP}^n$ such that

$$\forall x \in \tilde{M}, \forall \gamma \in \pi_1(M), f(\gamma x) = \rho(\gamma)f(x).$$

In this paper, since we will consider projective structures deformed from hyperbolic structures, all holonomy representations will lift to $SL(n+1, \mathbb{R})$. An \mathbb{RP}^n -structure is convex if the developing map is a homeomorphism onto a convex domain in \mathbb{RP}^n . It is *properly convex* if the domain is included in a compact convex set of an affine chart, *strictly convex* if the convex set is strictly convex. Surprisingly, while many people are working on global structures of the Hitchin component, it seems that the local structure has been neglected. This is the starting point of this article. We shall first compute the cohomology $H^1(\pi_1(M), \rho, \mathfrak{sl}(n+1, \mathbb{R}))$ of a Fuchsian point ρ which corresponds to a hyperbolic structure $\pi_1(M) \rightarrow SO(n, 1) \subset SL(n+1, \mathbb{R})$. See Section 2.1 for details. The cohomology is described in terms of quadratic and cubic forms. We shall use the result of Labourie [9] where he proved that a convex projective flat structure on M defines a Riemannian metric and a cubic form on M .

Theorem 1.1. *Let $\rho : \pi_1(M) \rightarrow SO(n, 1) \subset SL(n+1, \mathbb{R})$, $n > 2$, be a representation defining a real hyperbolic structure on a closed n -manifold M . Let $\alpha \in H^1(\pi_1(M), \rho, \mathfrak{sl}(n+1, \mathbb{R}))$. Then α is represented by a cubic form.*

¹2000 *Mathematics Subject Classification.* 51M10, 57S25.

²*Key words and phrases.* Zariski tangent space, real projective structure, Weil's local rigidity.

³Research partially supported by STINT-NRF grant (2011-0031291). Research by G. Zhang is supported partially by the Swedish Science Council (VR). I. Kim gratefully acknowledges the partial support of grant (NRF-2014R1A2A2A01005574) and a warm support of Chalmers University of Technology during his stay.

For $n = 2$, an element in $H^1(\pi_1(M), \rho, \mathfrak{sl}(3, \mathbb{R}))$ is represented by a sum of a quadratic form and a cubic form when ρ defines a convex real projective structure. This is due to [10, 9]. In this case both the global and local structures have been studied intensively. Recently [8] we have been able to construct mapping class group invariant Kähler metric on the Hitchin component of $SL(3, \mathbb{R})$, this is also part of our motivation of the present paper. We also mention that Labourie [9] has computed the cohomology $H^1(\pi_1(M), \mathbb{R}^3)$ where $\pi_1(M)$ acts on \mathbb{R}^3 through ρ and the defining representation of $SL(3, \mathbb{R})$.

As a corollary of our technique, we show the Weil's local rigidity theorem for uniform real hyperbolic lattices for dimension > 2 .

Theorem 1.2. *Let $M = \Gamma \backslash SO^0(n, 1) / SO(n)$ be a compact hyperbolic manifold. If $n > 2$ then $H^1(\Gamma, \mathfrak{so}(n, 1)) = 0$.*

Acknowledgement We are grateful for the anonymous referee for the careful reading of an earlier version of this paper and many useful comments.

2. TANGENT SPACE AT FUCHSIAN LOCUS OF CONVEX PROJECTIVE STRUCTURES ON A MANIFOLD M

2.1. Projective structure. The notion of projective structures can be formulated in terms of a flat connection as follows, see [9] for details. Consider a trivial bundle $E = M \times \mathbb{R}^{n+1}$ where M is an n -dimensional manifold. Let ω be a volume form on \mathbb{R}^{n+1} and let ∇ be a flat connection on E preserving ω . Let ρ be the holonomy representation of ∇ . A section u of E is identified with a ρ -equivariant map from \tilde{M} to \mathbb{R}^{n+1} . A section u is said to be ∇ -immersed if the n -form Ω_u defined by

$$\Omega_u(X_1, \dots, X_n) = \omega(\nabla_{X_1} u, \dots, \nabla_{X_n} u, u)$$

is non-degenerate. Then u is ∇ -immersed if it is a non-vanishing section and if the associated ρ -equivariant map $[u]$ from \tilde{M} to \mathbb{RP}^n is an immersion.

Hence it follows that such a pair (∇, u) gives rise to a flat projective structure. Labourie reformulated this as a pair of torsion free connection ∇^T on M with a symmetric 2-tensor h on M as follows. One can associate a connection ∇ on $E = TM \oplus \mathcal{L}$, where \mathcal{L} is a trivial bundle $M \times \mathbb{R}$, defined by

$$(2.1) \quad \nabla_X \begin{pmatrix} Y \\ \lambda \end{pmatrix} = \begin{pmatrix} \nabla_X^T Y & X \\ h(X, \cdot) & L_X \end{pmatrix} \begin{pmatrix} Y \\ \lambda \end{pmatrix} = \begin{pmatrix} \nabla_X^T Y + \lambda X \\ h(X, Y) + L_X(\lambda) \end{pmatrix}.$$

Here $L_X(\lambda) = X\lambda$ denotes the differentiation. Labourie [9] showed that if ∇ is flat and ∇^T preserves the volume form defined by h , then ∇ gives rise to a flat projective structure. He further showed that h is positive definite if the structure is properly convex. We will use this final form of projective structure in this paper to carry out the explicit calculations.

2.2. Tangent space of convex projective structures. Let M be an n -dimensional manifold and $\Gamma = \pi_1(M)$ its fundamental group. Let ρ be a representation of Γ into $SL(n+1, \mathbb{R})$ defining a convex projective structure on M . There is a flat connection ∇ on the associated \mathbb{R}^{n+1} -bundle E preserving a volume form as in the previous section.

The flat connection ∇ on E defines also a connection on the dual bundle E^* , $\mathfrak{gl}(E) = E \otimes E^*$, the bundle of endomorphisms of E , and further on $\mathfrak{sl}(E)$, the trace free endomorphisms, since ∇ preserve the volume form on E , by the Leibniz rule and the commutative relation with the contractions.

Write temporarily \mathfrak{F} for any of these flat bundles with fiber space F . We fix convention that if $\mathfrak{F} = E^*$ or $\mathfrak{sl}(E) \subset \mathfrak{gl}(E) = E \otimes E^*$, we write then a F -valued one-form α as $\alpha : (X, y) \rightarrow \alpha(X)y$ with the first argument being tangent vector and the second argument being element of F . For conceptual clarity we recall that given ∇^E on E the connection ∇^{E^*} on E^* -valued sections is defined by the equation

$$X(\alpha(y)) = (\nabla_X^{E^*} \alpha)(y) + \alpha(\nabla_X^E y);$$

whereas the connection on sections of $\mathfrak{sl}(E)$ is defined by

$$(2.2) \quad \nabla_X^E(\alpha(y)) = (\nabla_X^{\mathfrak{sl}} \alpha)(y) + \alpha(\nabla_X^E y).$$

We shall abbreviate them all as ∇_X .

The flat connection ∇ on E as well as its induced connection ∇^T induces exterior differentiation d^∇ and d^{∇^T} on 1-forms, locally defined as

$$d^\nabla(\sum \omega_i dx_i) = \sum \nabla \omega_i dx_i,$$

where ω_i is a local section. For notational convenience we shall write all of them just as d^∇ ; no confusion would arise as it will be clear from the context which sections they are acting on. We will freely write a $(0, 2)$ -tensor as $\alpha(X)Y = \alpha(X, Y)$. We shall need the following formula for the exterior differentiation on a $End(E)$ -valued one-form:

$$\begin{aligned} & (d^\nabla \alpha)(X, Z)y \\ &= (\nabla_X \alpha)(Z)y - (\nabla_Z \alpha)(X)y \\ (2.3) \quad &= \nabla_X(\alpha(Z)y) - \alpha(\nabla_X Z)y - \alpha(Z)(\nabla_X y) \\ &\quad - (\nabla_Z(\alpha(X)y) - \alpha(\nabla_Z X)y - \alpha(X)(\nabla_Z y)) \\ &= \nabla_X(\alpha(Z)y) - \alpha(Z)(\nabla_X y) - (\nabla_Z(\alpha(X)y) - \alpha(X)(\nabla_Z y)) - \alpha([X, Z])y \end{aligned}$$

since ∇ is flat, in particular ∇^T is torsion free.

We shall describe the cohomology in terms of some symmetry conditions of certain tensors. Let g be a Riemannian metric on M with the Levi-Civita connection ∇^g , the corresponding exterior differentiation being denoted by d^g . We consider following three conditions for $(0, 2)$ -tensors α , in which case α is a quadratic form:

- (q1): α is symmetric,
- (q2): trace-free with respect to g ,
- (q3): α is d^{∇^g} closed, $d^{\nabla^g} \alpha = 0$,

and the following four conditions for a $End(TM)$ -valued one-form, in which case α is a cubic form:

- (c1): $\alpha(X)Y$ is symmetric in X, Y , $\alpha(X)Y = \alpha(Y)X$,
- (c2): $\alpha(X)$ is symmetric with respect to g , $\alpha(X)^* = \alpha(X)$,
- (c3): α is d^{∇^g} closed, $d^{\nabla^g}\alpha = 0$, equivalently the cubic form $g(\alpha(X)Y, Z)$ is closed,
- (c4): $\alpha(X)$ is trace-free.

The following theorem is proved in [9, Theorems 3.2.1 & Proposition 4.2.3].

Theorem 2.1. *Suppose M admits a properly convex projective structure. Then there is splitting of the bundle*

$$(2.4) \quad E = TM \oplus \mathcal{L}$$

where \mathcal{L} is a trivial bundle $M \times \mathbb{R}$ and a Riemannian metric g on M such that the flat connection ∇ on E is given by

$$(2.5) \quad \nabla_X \begin{pmatrix} Y \\ \lambda \end{pmatrix} = \begin{pmatrix} \nabla_X^T Y & X \\ g(X, \cdot) & L_X \end{pmatrix} \begin{pmatrix} Y \\ \lambda \end{pmatrix} = \begin{pmatrix} \nabla_X^T Y + \lambda X \\ g(X, Y) + L_X(\lambda) \end{pmatrix}.$$

Here $L_X(\lambda) = X\lambda$ is the differentiation, ∇^T is a torsion-free connection on TM preserving the volume form of the Riemannian metric g and ∇^g is the Levi-civita connection of g , such that the tensor c , defined by $g((\nabla_X^T - \nabla_X^g)Y, Z) = c(X, Y, Z)$, satisfies the condition (c1-c4).

In other words such a flat connection ∇ determines a metric g on TM and thus on the bundle $TM \oplus \mathcal{L}$,

$$g \oplus |\cdot|^2 : (X, \lambda) \mapsto g(X, X) + \lambda^2$$

such that the connection ∇ takes the form

$$\nabla_X = \begin{bmatrix} \nabla_X^g & 0 \\ 0 & L_X \end{bmatrix} + \begin{bmatrix} Q(X) & X \\ X_g^\sharp & 0 \end{bmatrix}$$

where $\nabla_X^T - \nabla_X^g = Q(X)$, hence the first part is a skew-symmetric (i.e. orthogonal) connection and the second part is a symmetric form.

We shall now compute the tangent space of the deformation space at Fuchsian locus induced from the natural inclusion $SO(n, 1) \subset SL(n+1, \mathbb{R})$. A hyperbolic n -manifold can be viewed as a real projective manifold and it can be deformed inside the convex real projective structures for any dimension $n \geq 2$. The component containing the hyperbolic structures constitutes the strictly convex real projective structures [2, 3, 6]. Indeed, any convex real projective structures whose holonomy group is not contained in $SO(n, 1)$ has a Zariski dense holonomy group in $SL(n+1, \mathbb{R})$, see [1].

Hence from now on, we shall assume $Q = 0$ and then g is a hyperbolic metric on M with constant curvature -1 . All the covariant differentiations below will be the one induced by ∇^g .

Theorem 2.2. *Let $\rho : \pi_1(M) \rightarrow SO(n, 1) \subset SL(n + 1, \mathbb{R})$, $n > 2$ be a representation defining a hyperbolic structure on the compact n -manifold M . Let g be the hyperbolic metric determined by ∇ . Then there exists an injective map from $H^1(\pi_1(M), \rho, \mathfrak{sl}(n + 1, \mathbb{R}))$ into the space of cubic forms satisfying (c1-c4).*

The proof will be divided into several steps.

Let α be a one-form representing an element of $H^1(\pi_1(M), \rho, \mathfrak{sl}(n + 1, \mathbb{R}))$, realized as an element of $\Omega^1(M, \mathfrak{sl}(E))$. Write it as

$$X \mapsto \alpha(X) = \begin{bmatrix} A(X) & B(X) \\ C(X) & D(X) \end{bmatrix} \in \mathfrak{sl}(E),$$

under the splitting (2.4), where $A \in \Omega^1(M, \text{End}(TM))$, $B \in \Omega^1(M, TM)$, $C \in C^\infty(M, T^*M \otimes T^*M)$ and $X \mapsto D(X) = -\text{tr } A(X)$ is a one-form.

We first observe for the covariant differentiation of a section of $\mathfrak{sl}(E)$

$$u = \begin{pmatrix} a & b \\ c & e \end{pmatrix}$$

is the one-form

$$(2.6) \quad X \mapsto \nabla_X u = \begin{pmatrix} (\nabla_X^g a) \cdot + c(\cdot)X - g(X, \cdot)b & \nabla_X^g b + eX - a(X) \\ g(X, a \cdot) + (\nabla_X^g c)(\cdot) - e(X, \cdot) & (Xe) + g(X, b) - c(X) \end{pmatrix},$$

acting on a section $y = (Y, \lambda)$, where the dots denote the variable Y . These are exact forms.

Lemma 2.3. *Up to exact forms we can assume*

$$B = 0, D = 0.$$

Proof. We prove first that we can choose u so that $\alpha + \nabla u$ has its entries $B(X) = bX$, $D = 0$ where b is a scalar function. Indeed we choose

$$u = \begin{pmatrix} a & 0 \\ c & e \end{pmatrix},$$

$a = B$, c the one-form $c = de + D$. The form $\alpha + \nabla u$ then has the desired form:

$$\begin{aligned} X \mapsto \alpha(X) + \nabla_X u &= \begin{bmatrix} A(X) + (\nabla_X^g B) \cdot + c(\cdot)X & B(X) + eX - B(X) \\ C(X) + g(X, a \cdot) + (\nabla_X^g c) \cdot - g(X, \cdot)e & D(X) + L_X(e) - c(X) \end{bmatrix}, \\ &= \begin{bmatrix} A'(X) & eX \\ C'(X) & 0 \end{bmatrix} \end{aligned}$$

as claimed. We write the new form $\alpha + \nabla u$ as α with its entries $B = b\text{Id}$, $D = 0$.

Next we take

$$v = \begin{pmatrix} a_1 \text{Id} & 0 \\ c_1 & -na_1 \end{pmatrix}, \quad a_1 := \frac{1}{n+1}b, \quad c_1 := -nda_1 = -\frac{n}{n+1}db,$$

the same calculation above shows that $\alpha + \nabla v$ has its $B = 0$, $D = 0$. \square

We shall need the precise formula of the diagonal part in the lower triangular form, keeping track of the computations we find $\alpha + \nabla(u + v)$ has the form

$$(2.7) \quad \begin{pmatrix} A(X) + \nabla_X^g B + (Xb)I + fX & 0 \\ * & 0 \end{pmatrix}, \quad b = -\frac{1}{n+1} \operatorname{tr} B, \quad f = db$$

Lemma 2.4. *Let α be lower triangular with $B = 0$, $D = 0$. The covariant derivatives $d^g A$ and $d^g C$ are related to A and C by*

$$(2.8) \quad \boxed{0 = (d^g A)(X, Z)(Y) + (C(Z)Y)X - (C(X)Y)Z}$$

and

$$(2.9) \quad \boxed{0 = g(X, A(Z)Y) - g(Z, A(X)Y) + (d^g C)(X, Z)(Y),}$$

and there hold the symmetric relations:

$$(2.10) \quad \boxed{A(X)Z = A(Z)X},$$

$$(2.11) \quad \boxed{C(X)Z = C(Z)X}.$$

Proof. We write an arbitrary section y of E as $y = (Y, \lambda) = (y_T, y_n)$, the tangential and respectively normal component. To write down the condition on the closedness $d^\nabla \alpha(X, Z) = 0$ in terms of the components A, C , we recall (2.3). The condition $d^\nabla \alpha(X, Z) = 0$ is then

$$(2.12) \quad \begin{aligned} 0 &= d^\nabla \alpha(X, Z)y \\ &= v(X, Z; y) - v(Z, X; y) - \alpha([X, Z])y \\ &= v(X, Z; y) - v(Z, X; y) - \begin{pmatrix} A([X, Z])Y \\ C([X, Z])Y \end{pmatrix}. \end{aligned}$$

where

$$\begin{aligned} v(X, Z; y) &:= \nabla_X(\alpha(Z)y) - \alpha(Z)(\nabla_X y) = \\ &= \begin{pmatrix} \nabla_X^g(A(Z)Y) + (C(Z)Y)X - A(Z)(\nabla_X^g Y + \lambda X) \\ g(X, A(Z)Y) + X(C(Z)Y) - C(Z)(\nabla_X^g Y + \lambda X) \end{pmatrix}. \end{aligned}$$

Here we have used the fact that $B = 0$, $D = 0$ in the computations.

Recall that $d^g A = d^{\nabla^g} A$ is defined by

$$(2.13) \quad \begin{aligned} (d^g A)(X, Z)(Y) &= \\ &= \nabla_X^g(A(Z)Y) - A(Z)(\nabla_X^g Y) - (\nabla_Z^g(A(X)Y) - A(X)(\nabla_Z^g Y)) - A([X, Z])Y \end{aligned}$$

which is a well-defined $\operatorname{End}(TM)$ -valued 2-form, and

$$(2.14) \quad \begin{aligned} (d^g C)(X, Z)(Y) &= \\ &= L_X(C(Z)Y) - C(Z)(\nabla_X^g Y) - (L_Z(C(X)Y) - C(X)(\nabla_Z^g Y)) \\ &\quad - C([X, Z])Y \end{aligned}$$

is the Riemannian exterior differential of the form C , and is an element of $\Omega^2 \otimes \Omega$. The first two equations (2.8) -(2.9) are obtained from (2.12) by putting $y = (Y, 0)$.

Correspondingly we have, taking $y = (0, 1)$,

$$(2.15) \quad A(X)Z - A(Z)X = 0$$

and

$$(2.16) \quad C(X)Z - C(Z)X = 0,$$

resulting the symmetric relations (2.10)- (2.11). \square

Let α_0 be the bilinear form

$$\alpha_0(X, W) = C(X, W) := C(X)W$$

and α_1 the $End(TM)$ -valued one-form

$$\alpha_1(Y)X := \frac{1}{2}(A(Y)X + A_g^*(Y)X).$$

Then $\alpha_0(X, W)$ is symmetric in X and W , hence it satisfies (q1). $\alpha_1(Y)$ is symmetric with respect to g and trace free, since $\alpha(Y) \in \mathfrak{sl}(E)$, $0 = \text{tr } \alpha(Y) = \text{tr } A(Y) + D(Y) = \text{tr } A(Y)$, hence it satisfies the conditions (c1) and (c4).

Lemma 2.5. *Let α be of the above form with $B = 0, D = 0$. Then we have $C = 0$ and $\alpha_0 = 0$ and α_1 satisfies the conditions (c1)-(c4) for $n > 2$.*

Proof. The equation (2.9) combined with (2.10) implies that

$$g(X, A(Y)Z) - g(Z, A(Y)X) + (d^g C)(X, Z)Y = 0.$$

In other words

$$(2.17) \quad (A_g^*(Y) - A(Y))X = -((d^g C)(X, \cdot)(Y))_g^\sharp$$

where the lowering of the index in the right hand side is with respect to the second variable. Since g is parallell with respect to ∇^g then $((d^g C)(X, \cdot)(Y))_g^\sharp$ is an exact $End(TM)$ -valued one form. This is not obvious and requires proof. Indeed, let C^b be the $End(TM)$ -valued 0-form,

$$C^b(X) = \sum_i C(X, Z_i)Z_i$$

where $\{Z_i\}$ is a local orthonormal frame of TM . We claim that

$$(2.18) \quad ((d^g C)(X, \cdot)(Y))_g^\sharp = (d_Y^g C^b)(X).$$

By definition we have the identity section $\text{Id} = \sum_i Z_i \otimes Z_i^\sharp$ and,

$$0 = \nabla_Y^g \text{Id} = \sum_i (\nabla_Y^g Z_i \otimes Z_i^\sharp + Z_i \otimes \nabla_Y^g (Z_i^\sharp)) = \sum_i \nabla_Y^g Z_i \otimes Z_i^\sharp + \sum_i Z_i \otimes (\nabla_Y^g Z_i)^\sharp,$$

namely, for any Z

$$(2.19) \quad 0 = \sum_i g(Z_i, Z) \nabla_Y^g Z_i + \sum_i g(\nabla_Y^g Z_i, Z) Z_i.$$

Here we have used the fact that ∇_Y^g commutes with \sharp , $\nabla_Y^g(Z_i^\sharp) = (\nabla_Y^g Z_i)^\sharp$. By definition, LHS of (2.18) is

$$\begin{aligned} LHS &= \sum_i ((d^g C)(X, Z_i)(Y))Z_i = \sum_i ((\nabla_Y^g C)(X, Z_i))Z_i \\ &= \sum_i Y(C(X, Z_i))Z_i - \sum_i C(\nabla_Y^g X, Z_i)Z_i - \sum_i C(X, \nabla_Y^g Z_i)Z_i. \end{aligned}$$

Here $C \in C^\infty(M, T^*M \otimes T^*M)$ is a zero form, hence $(d^g C)(X, Z_i)(Y) = (\nabla_Y^g C)(X, Z_i)$. On the other hand,

$$\begin{aligned} RHS &= (\nabla_Y^g C^\flat)(X) = \nabla_Y^g(C^\flat(X)) - C^\flat(\nabla_Y^g X) \\ &= \nabla_Y^g \left(\sum_i C(X, Z_i)Z_i \right) - \sum_i C(\nabla_Y^g X, Z_i)Z_i \\ &= \sum_i Y(C(X, Z_i))Z_i + \sum_i C(X, Z_i)\nabla_Y^g Z_i - \sum_i C(\nabla_Y^g X, Z_i)Z_i. \end{aligned}$$

To treat the second term we compute its inner product with any Z ; it is

$$\begin{aligned} \sum_i C(X, Z_i)g(\nabla_Y^g Z_i, Z) &= C(X, \sum_i g(\nabla_Y^g Z_i, Z)Z_i) \\ &= -C(X, \sum_i g(Z_i, Z)\nabla_Y^g Z_i) \\ &= -g\left(\sum_i C(X, \nabla_Y^g Z_i)Z_i, Z\right) \end{aligned}$$

where the second equality is by (2.19). Hence

$$\sum_i (C(X, Z_i))\nabla_Y^g Z_i = - \sum_i C(X, \nabla_Y^g Z_i)Z_i,$$

proving $RHS = LHS$ and hence confirming (2.18).

The form α_1 can now be written as

$$\alpha_1(Y) = \frac{1}{2}(A_g^*(Y) + A(Y)) = A(Y) + \frac{1}{2}(A_g^*(Y) - A(Y))$$

with the second term $\frac{1}{2}(A_g^*(Y) - A(Y))$ being exact, which implies that $d^g \alpha_1 = d^g A$. The equation (2.8) can now be written as

$$(2.20) \quad 0 = (d^g \alpha_1)(X, Z)(Y) + (C(Z)Y)X - (C(X)Y)Z$$

where $\alpha_1(X)$ is trace-free and symmetric with respect to g . This in turn implies that the map $Y \rightarrow C(Z, Y)X - C(X, Y)Z$ is symmetric,

$$\begin{aligned} (2.21) \quad &g(C(Z, Y)X - C(X, Y)Z, W) \\ &= g(Y, C(Z, W)X - C(X, W)Z). \end{aligned}$$

Let $\{Z_i\}$ be an orthonormal basis, $Y = Z = Z_i$, and summing over i , we get

$$(2.22) \quad (\text{tr}_g C)g(X, W) + (n-2)C(X, W) = 0.$$

Taking again the trace we find

$$(2.23) \quad \text{tr}_g C = 0.$$

Hence α_0 satisfies (q2).

Substituting this into the previous formula we get

$$(n - 2)(C(X, W)) = 0,$$

and consequently

$$(2.24) \quad C(X, W) = 0$$

if $n > 2$.

For $n > 2$ it follows from (2.24) and (2.17) that

$$(2.25) \quad A_g^*(Y) = A(Y)$$

i.e, $A(Y)$ is symmetric with respect to g . Consequently

$$\alpha_1(Y) = A(Y) = A_g^*(Y).$$

Hence $\alpha_1(Y)X = A(Y)X = A(X)Y = \alpha_1(X)(Y)$ by Equation (2.10). This proves that α_1 satisfies (c1)-(c2), (c4) for $n > 2$. The equation (2.20) combined with $C = 0$ implies further $d^g \alpha_1 = 0$. Hence α satisfies all the conditions to be a cubic form. \square

We prove now that the map from α to the cubic form in Lemma 2.5 is injective.

Lemma 2.6. *Let $n > 2$ and suppose $\alpha \in H^1(\pi_1(M), \rho, \mathfrak{sl}(n + 1, \mathbb{R}))$. Then the map $\alpha \mapsto \alpha_1$ from $H^1(\pi_1(M), \rho, \mathfrak{sl}(n + 1, \mathbb{R}))$ to the cubic form α_1 is injective.*

Proof. In Lemma 2.5, we showed that if α is represented by a $\mathfrak{sl}(E)$ -valued one form with $B = D = 0$, then $C = 0$ and the associated cubic form is $\alpha_1(Y) = A(Y)$. Hence if the associated cubic form vanishes, $A = 0$. This implies that α is represented by an exact 1-form by Equation (2.7), hence it is a zero element in the cohomology. \square

This finishes the proof of Theorem 2.2.

Now we prove the Weil's local rigidity theorem, $H^1(\Gamma, \mathfrak{so}(n, 1)) = 0$, $n > 2$, [12, Chapter VII], [13] as an application of our technique. For $n = 2$, it is well-known that the cohomology $H^1(\Gamma, \mathfrak{so}(2, 1))$ is determined by quadratic forms satisfying (q1-q3), namely real part of holomorphic quadratic forms; see e.g. [4].

Theorem 2.7. *Let $M = \Gamma \backslash SO^0(n, 1) / SO(n)$ be a compact hyperbolic manifold.*

- (1) *If $n > 2$ then $H^1(\Gamma, \mathfrak{so}(n, 1)) = 0$.*
- (2) *If $n = 2$ then $H^1(\Gamma, \mathfrak{so}(n, 1))$ is given by the space of quadratic forms satisfying (q1-q3).*

Proof. Let α represent an element in $H^1(\Gamma, \mathfrak{so}(n, 1))$ viewed as an element in $\Omega^1(M, \mathfrak{so}(n, 1))$.

The elements in $\mathfrak{so}(n, 1)$ are of the form $\begin{bmatrix} a & b \\ c & 0 \end{bmatrix}$ with $a = -a^*$ and $b = c^T$ with respect to the Euclidean product in \mathbb{R}^n as a subspace of the Lorentz space \mathbb{R}^{n+1} . The 1-form α takes then the form

$$\alpha = \begin{bmatrix} A & B \\ C & 0 \end{bmatrix}$$

with $g(A(X)Y, Z) = -g(Y, A(X)Z)$, $g(B(X), Y) = C(X)(Y) = C(X, Y)$, where g is the given hyperbolic metric on M . The $(1, 1)$ -entry A of α is skew-symmetric. Now from Equations (2.3), (2.2) and the fact that $\nabla^T = \nabla^g$ for hyperbolic manifold, $(1, 1)$ -entry of the condition $d^g \alpha = 0$ is

$$(2.26) \quad (d^g A)(X, Z) + X \otimes C(Z) - Z \otimes C(X) + B(X) \otimes Z^\sharp - B(Z) \otimes X^\sharp = 0$$

as two form acting on (X, Z) . Here $X \otimes Z^\sharp$ is the rank-one map $Y \mapsto g(Y, Z)X$. We shall also need some Hodge theory. Equip $\mathfrak{so}(n, 1)$ with the $SO(n)$ -invariant positive inner product induced from the standard Euclidean inner product in \mathbb{R}^{n+1} , $(y, y)_E = \|Y\|_g^2 + \lambda^2$, $y = (Y, \lambda)$. Take a harmonic one form representing α . Then the cohomology class α satisfies also the coboundary condition $\nabla^* \alpha = 0$. To write down the formula for ∇^* we observe that

$$\nabla_X = \begin{bmatrix} \nabla_X^g & 0 \\ 0 & L_X \end{bmatrix} + \begin{bmatrix} 0 & X \\ X^\sharp & 0 \end{bmatrix}$$

is a sum of two terms, the first preserving the Euclidean inner product $(y, y)_E$, whose adjoint can be found by standard formula (see e.g. [11, p.2]), whereas the second part is self-adjoint. Thus $-\nabla^* \alpha$ is given by

$$\sum_j (\delta_{X_j} \alpha)(X_j)$$

where

$$\delta_X = \begin{pmatrix} \nabla_X^g & -X \\ -X^\sharp & L_X \end{pmatrix}.$$

More precisely, $(\delta_X \alpha)(Z)$ is given, for any testing section $y = (Y, \lambda)$, by the Leibniz rule

$$\begin{aligned} & (\delta_X \alpha)(Z)y \\ &= \begin{pmatrix} \nabla_X^g & -X \\ -X^\sharp & L_X \end{pmatrix} (\alpha(Z)y) - \alpha(Z) \left(\begin{pmatrix} \nabla_X^g & -X \\ -X^\sharp & L_X \end{pmatrix} y \right) - \alpha(\nabla_X^g Z)y. \end{aligned}$$

(The sum $\sum_j (\delta_{X_j} \alpha)(X_j)$ is well-defined but not the individual terms.) When acting on the section $y = (0, 1)$ we find

$$(2.27) \quad \sum_j (A(X_j)X_j + (\nabla_{X_j}^g B)(X_j)) = 0.$$

It now follows from Equation (2.7), Lemmas 2.3 and 2.5 (keeping track of the change of forms) that $A_1(X) := A(X) + \nabla_X^g B + (Xb)I + fX$ is symmetric and trace-free and satisfies the condition (c1-c4). Note here while performing computations as in Lemmas 2.3-2.5 we use forms u with values in $\mathfrak{sl}(n+1)$ instead of in $\mathfrak{so}(n, 1)$, however all we need

is that $d^\nabla d^\nabla = 0$, i.e. we will show that α vanishes identically. The trace free condition and

$$A(X)Y + \nabla_X^g B(Y) + (Xb)Y + f(Y)X = A(Y)X + \nabla_Y^g B(X) + (Yb)X + f(X)Y$$

imply that the map $Z \rightarrow A_1(Z)Y = A(Z)Y + (\nabla_Z B)(Y) + (Zb)Y + f(Y)Z$ is trace free. The symmetric relation implies

$$\begin{aligned} & g(A(Z)Y + (\nabla_Z B)(Y) + (Zb)Y + f(Y)Z, W) \\ &= g(A(Z)W + (\nabla_Z B)(W) + (Zb)W + f(W)Z, Y). \end{aligned}$$

We take $\{Z_j\}$ a local orthonormal frame and put $Z = W = Z_j$ in the above equation. Summing over j we find, in view of (2.27) that the right hand side is

$$RHS = \sum_j g(A(Z_j)Z_j + (\nabla_{Z_j} B)(Z_j) + (Z_j b)Z_j + f(Z_j)Z_j, Y) = (Yb) + f(Y)$$

and

$$LHS = \text{tr}(A_1(\cdot)Y) = 0.$$

Namely the one-form $Yb + f(Y) = 0$. But $f(Y) = (db)(Y) = Yb$ by (2.7), so $0 = 2f(Y)$, and $db = f = 0$. This implies in turn that

$$A_1(X) = A(X) + \nabla_X^g B + (Xb)I + fX = A(X) + \nabla_X^g B$$

is symmetric. We write $B = B^0 + B^1$, the symmetric and respectively the skew symmetric part of B . Since A is skew symmetric, the skew symmetric part of $A_1(X)$ must vanish, that is

$$(2.28) \quad A(X) + \nabla_X^g B^1 = 0.$$

This implies in turn $A(X) = -\nabla_X B^1$ is exact. Thus $d^g A = 0$, and the relation (2.26) becomes

$$X \otimes C(Z) - Z \otimes C(X) + B(X) \otimes Z^\sharp - B(Z) \otimes X^\sharp = 0.$$

Let $\{Y_i\}$ be an orthonormal basis, put $Z = Y_i$ and let the above act on Y_j . Taking the sum and using $B^t = C$ we find as in the proof of Lemma 2.5, that

$$\sum_i g(X \otimes C(Y_i, Y_i) - Y_i \otimes C(X, Y_i) + B(X)g(Y_i, Y_i) - B(Y_i)g(X, Y_i), W) = 0$$

i.e.

$$\text{tr}_g Cg(X, W) - C(X, W) + nC(X, W) - C(X, W) = 0.$$

But the same proof above implies that

$$(2.29) \quad \text{tr}_g C = 0, \quad (n-2)C(X, W) = 0.$$

Now let $n > 2$. Thus $C(X, W) = 0$. Then $B = C^t = 0$ and using the symmetric condition on $A_1 = A$ we find that A is symmetric and thus $A = 0$. This proves (1).

Let $n = 2$. We consider the $\mathfrak{so}(2, 1)$ -valued section

$$u = \begin{pmatrix} B^1 & 0 \\ 0 & 0 \end{pmatrix}.$$

Using the formulas (2.6) and (2.28) we find

$$\alpha + \nabla u : X \mapsto \alpha(X) + \nabla_X u = \begin{bmatrix} 0 & B^0 \\ C^0 & 0 \end{bmatrix}$$

where C^0 is the symmetric part of C and $(B^0)^t = C^0$. So replacing α by $\alpha + \nabla u$ we may assume that $A = 0$, B is symmetric and $B^t = C$. A direct calculation using (2.12) and $y = (Y, 0)$ gives

$$\begin{aligned} 0 &= d^\nabla \alpha(X, Z)Y \\ &= \begin{bmatrix} (C(Z)Y)X - g(X, Y)B(Z) - (C(X)Y)Z + g(Z, Y)B(X) \\ X(C(Z)Y) - C(Z)\nabla_X^g Y - Z(C(X)Y) + C(X)\nabla_Z^g Y - C([X, Z])Y \end{bmatrix}. \end{aligned}$$

But by the formula (2.14)

$$X(C(Z)Y) - C(Z)\nabla_X^g Y - Z(C(X)Y) + C(X)\nabla_Z^g Y - C([X, Z])Y = (d^g C)(X, Z)Y$$

Hence $d^\nabla \alpha = 0$ gives $d^g C = d^g B = 0$. We have thus that B is symmetric, $\text{tr}_g B = \text{tr}_g C = 0$ by (2.29) and $d^g B = 0$, namely B satisfies (q1-q3).

This completes the proof. \square

Recall [5] that the Hitchin component, denoted by $\chi_H(\pi_1(S), SL(3, \mathbb{R}))$, is a connected component in the character variety $\text{Hom}(\pi_1(S), SL(3, \mathbb{R}))/SL(3, \mathbb{R})$ containing the realization of $\pi_1(S)$ as a subgroup of $SL(2, \mathbb{R})$ composed with the irreducible representation of $SL(2, \mathbb{R})$ on \mathbb{R}^3 . The tangent space of $\chi_H(\pi_1(S), SL(3, \mathbb{R}))$ at these specific Fuchsian points can be obtained from the general theory in [5]. See also [7].

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